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Principles of General Covariance and the Role of Coordinates in General Relativity

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During the period 1 January to 1 July we continued our investigations on various aspects of the Principle of General Covariance both in classical and quantum physics. In particular we have obtained a formulation of the principle which reflects the topological properties of the space-time manifold. We have then had to define what is usually meant by a symmetry of a physical theory. This was done by dividing the geometrical objects which describe the trajectories of the system into two groups. One we call the dynamical objects whose equations of motion follow from a variational principle, the other, the absolute objects for which this is not true. The symmetry group of the theory is then that subgroup of arbitrary mappings of the space-time manifold which leaves invariant the absolute objects of the system. Our definition of a symmetry group is thus a geometrical definition and does not depend on any particular coordinitization of the manifold. A preliminary account of certain aspects of this work will appear in a forthcoming book Gravity and Relativity, Benjamin, New York, 1964.

In addition to the above we have also considered further the problem of the construction of Dirac brackets for a generally covariant theory. We have been able to show that when the reference frame is specified by means of coordinate conditions which fix it up to at most a finite number of parameters the Dirac brackets of the remaining field variables become well-defined.

We are also continuing our study of the Principle of General Covariance in a quantum field context but have no new results to report over and above those contained in the attached report.

Reference is also made to report, "Coordinate Conditions and Canonical Formalisms in Gravitational Theory," SIT-P96 forwarded 17 May 1963.

* Q-NUMBER COORDINATE TRANSFORMATIONS AND THE ORDERING PROBLEM IN GENERAL RELATIVITY

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Q-NUMBER COORDINATE TRANSFORMATIONS

The quantization of general relativity can be effected in either of two ways. One can destroy the covariance of the theory at the classical level by the introduction of coordinate conditions and consider only what remains of the metric to be represented by operators defined on a Hilbert space. Alternately, one can consider all parts of the metric to be operators defined on a linear vector space and only afterwards restrict the freedom of coordinate transformations by imposing coordinate conditions. The two procedures should of course lead to the same predictions and it is clear that they do. What is not clear is that one obtains equivalent theories starting from two essentially different sets of coordinate conditions.

Since in the first approach, which we shall call the Hilbert space quantization, the coordinate invariance was destroyed prior to quantization it is difficult to discuss the equivalence of quantized theories based on different coordinate conditions. In what we call the linear vector space quantization such is not the case since the coordinate conditions are introduced only after quantization and hence, in its initial stages, the theory still allows one to carry out coordinate transformations. However, here the difficulty arises that the transformation which effects the transition from one set of coordinate conditions to finitely different different set will depend upon the metric and hence will appear as a q-number transformation.

* Eastern Theoretical Physics Conference, University of Virginia October 26–27, 1962; Edited by M. E. Rose, Professor of Physics University of Virginia; Gordon and Breach, Science Publishers.

Ordinarily such a transformation would not preserve the basic commutation relations between the metric components and their canonical conjugates. We can circumvent this difficulty by dealing first with infinitesimal coordinate transformations generated by the constraints of the theory.[1)] Actually the transformations generated by the constraints are true coordinate transformations in the subspace of the linear vector space where the constraint equations are satisfied, i.e., in the subspace whose elements Ψ satisfy the equations

$$X_a \, (g_{\mu\nu}, \, p^{\mu\nu}) \, \Psi = 0 \qquad (1)$$

where the X_a are the constraints of the theory. In order that linear combinations of the constraints with q-number coefficients serve as generators of a group and hence allow the construction of finite coordinate transformations the commutator of two such generators must again be a generator. This will be the case only if the commutator of any two constraints is a linear combination of constraints with all of the coefficients in the linear combination standing to the left of the constraints. Thus we must have that

$$X_a \, X_\beta - X_\beta X_a = w_{a\beta}{}^{\delta} \, X_\delta \qquad (2)$$

with the w's all standing to the left of the X's on the right-hand side of this equation. This condition of course follows immediately from equation (1) as a consistency condition. However for our purpose it is important to see how this condition arises as a necessary condition for the construction of finite q-number coordinate transformations.

Since classically the Poisson bracket of any two constraints is a linear combination of constraints the satisfaction of the conditions (2) reduces to finding an ordering of factors in the classical expressions for the constraints. In general relativity the primary constraints are just

$$p^{o\mu} = 0$$

so that there is no ordering problem for them. The longitudinal constraints are of the form

$$\mathcal{H}_s = g_{ab,s}\, p^{ab} - 2(g_{as}\, p^{ab})_{,b} = 0 \tag{3}$$

where p^{ab} is momentum density conjugate to the spatial part of the metric g_{ab}. The Poisson bracket of two longitudinal constraints is again a linear combination of longitudinal constraints and, if we adopt the ordering indicated in eq. (3) with the p's standing to the right of the g's, so is the commutator. Since the coefficients of the linear combination are c-numbers the requirements (2) are automatically satisfied.

An ordering problem arises only with the Hamiltonian constraints which have the classical form

$$\mathcal{H}_L = \frac{1}{K}\,(g_{ra}g_{sb} - \tfrac{1}{2}\; g_{rs}g_{ab})\; p^{rs}p^{ab} - {}^3R\;(g_{ab}) = 0 \tag{4}$$

where K is the square root of the determinant of g_{ab} and ${}^3R$ is the curvature scaler formed from g_{ab} and its inverse. We must find an ordering for the first term which reproduces, as commutation relations, the classical Poisson bracket relations

$$(\mathcal{H}_L\,,\, \mathcal{H}'_L) = -\,\mathcal{H}_r\, e^{rs}\; \delta_{,s}(x - x') + \mathcal{H}'_r\; e'^{rs}\; \delta_{,s'}(x - x') \tag{5}$$

and

$$(\mathcal{H}_L\,,\, \mathcal{H}'_s) = \{\, \mathcal{H}_L\, \delta(x - x')\}_{,s} \tag{6}$$

Independent of what ordering we start with, it can always be brought into the form

$$\mathcal{H}_L = \frac{1}{K}\,(g_{ra}g_{sb} - \tfrac{1}{2}\, g_{rs}g_{ab})\, p^{rs}\, p^{ab} - {}^3R(g_{ab}) - i\hbar\; \delta(0)\; \frac{a}{K}\; g_{rs}\, p^{rs} - \hbar^2\, \delta^2(0)\; \frac{b}{K} \tag{7}$$

by making use of the commutation relations between the g's and p's. Here a and b are numerical constants which depend upon the initial ordering taken. There are a number of values of a and b, including zero, which lead to a reproduction of the relations (5). However, a rather laborious but straightforward computation shows that there are none which reproduce the relations (6). The consistency conditions (2) therefore cannot be satisfied for the case of general relativity.

While our conclusion does not mean that we cannot quantize general relativity making use of some particular set of coordinate conditions will lead to an inequivalent quantum theory. This in turn would mean that, in principle at least, it should be possible rule out all but one set of coordinate conditions on the basis of experimental evidence, a conclusion which is contrary to the spirit of the principle of general covariance.

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1) J. L. Anderson and P. G. Bergmann, Phys. Rev. 83, 1018 (1951)